PHYSICS. Nov. 16.

α, β : parameters

Good conductors : $\frac{\sigma}{\omega\varepsilon} >> 1$

$$\alpha \approx \beta \approx \sqrt{\frac{\omega\mu\sigma}{2}}$$

$$\eta_c : \text{complex impedance} = \sqrt{\frac{\mu}{\varepsilon}}\left(1 - j\frac{\sigma}{\omega\varepsilon}\right)^{-1/2} \simeq \sqrt{\frac{\mu}{\varepsilon}}(-j)^{-1/2}\left(\frac{\sigma}{\omega\varepsilon}\right)^{-1/2}$$

$$(-j) = e^{3\pi j/2} \Rightarrow (-j)^{1/2} = e^{3\pi j/4}$$

$$(-j)^{-1/2} = \frac{1}{(-j)^{1/2}} = e^{-3\pi j/4} = \frac{1}{\sqrt{2}}(1+j)$$

$$\eta_c = \sqrt{\frac{\mu}{\varepsilon}\frac{\varepsilon\omega}{\sigma}}\frac{1}{\sqrt{2}}(1+j) = \sqrt{\frac{\omega\mu}{2\sigma}}(1+j)$$

POWER PROPAGATION IN A WAVE.

Poynting vector (Electromagnetic power density)

For cultural purposes:

$$\vec{F} = q\vec{E} + q\vec{u}\times\vec{B}$$

$$dW = \vec{F}\cdot d\vec{l}$$
$$= q\vec{E}\cdot d\vec{l} + q\,d\vec{l}\cdot(\vec{u}\times\vec{B})$$

magnetic force does no work, only changes direction

$$u = \frac{d\vec{l}}{dt} \quad \therefore (\vec{u}\times\vec{B}) \text{ is } \perp \text{ to } d\vec{l} \Rightarrow d\vec{l}\cdot(\vec{u}\times\vec{B}) = 0$$

$$\therefore dW = q\vec{E}\cdot d\vec{l}$$

$$\frac{1}{q}dW = \vec{E}\cdot d\vec{l} = dV$$

For N particles: $\overbrace{\frac{dW}{dt}}^{\text{power}} = Nq\vec{E}\cdot\frac{d\vec{l}}{dt} = Nq\vec{E}\cdot\vec{u} = \vec{E}\cdot\underbrace{(Nq\vec{u})}_{\vec{J}_c}$

cond. current density

$$\frac{dW}{dt} = \vec{E}\cdot\vec{J}_c$$

$$\vec{E} : \left[\frac{V}{M}\right], \quad \vec{J}_c : \frac{[I]}{[m^2]}$$

$$\vec{\nabla} \times \vec{H} = \vec{J}_c + \frac{\partial \vec{D}}{\partial t} \rightarrow \vec{J}_c = \vec{\nabla} \times \vec{H} - \frac{\partial \vec{D}}{\partial t}$$

$$\frac{\partial w}{\partial t} = \vec{E}(\vec{\nabla} \times \vec{H}) - \vec{E}\left(\frac{\partial \vec{D}}{\partial t}\right)$$

$$\vec{\nabla} \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\vec{\nabla} \times \vec{A}) - \vec{A} \cdot (\vec{\nabla} \times \vec{B}) \Longleftarrow \text{identity.}$$

$$\vec{E} \cdot (\vec{\nabla} \times \vec{H}) = \vec{H}(\vec{\nabla} \times \vec{E}) + \vec{\nabla} \cdot (\vec{H} \times \vec{E})$$

$$\vec{\nabla} \times \vec{E} = \frac{-\partial \vec{B}}{\partial t}$$

$$\frac{dw}{dt} = \vec{E} \cdot \vec{J}_c = -\vec{H}\left(\frac{\partial \vec{B}}{\partial t}\right) - \vec{\nabla} \cdot (\vec{E} \times \vec{H}) - \vec{E} \cdot \left(\frac{\partial \vec{D}}{\partial t}\right)$$

$$\vec{B} = \mu \vec{H}$$
$$\vec{D} = \varepsilon \vec{E}$$

$$\Rightarrow \frac{dw}{dt} = \vec{E} \cdot \vec{J}_c = -\frac{\mu}{2} \frac{\partial}{\partial t}(\vec{H} \cdot \vec{H}) - \frac{1}{2} \varepsilon \frac{\partial}{\partial t}(\vec{E} \cdot \vec{E}) - \vec{\nabla} \cdot (\vec{E} \times \vec{H})$$

Integrate over volume

$$\underbrace{\int_V (\vec{E} \cdot \vec{J}_c)\, dv}_{\text{power dissipated in the volume}} = -\underbrace{\int_V (\vec{\nabla} \cdot (\vec{E} \times \vec{H}))\, dv}_{-\int_S (\vec{E} \times \vec{H}) \cdot d\vec{s}} - \frac{\partial}{\partial t} \int_V \Big(\underbrace{\frac{\mu}{2}(\vec{H} \cdot \vec{H}) + \frac{\varepsilon}{2}(\vec{E} \cdot \vec{E})}_{\text{energy in fields}}\Big)\, dv$$

$$-\int_S (\vec{E} \times \vec{H}) \cdot d\vec{s}$$

$\hookrightarrow$ Flux of energy through the surface that bounds the volume.

$\vec{E} \times \vec{H} = \vec{P}$: Poynting vector
$\hookrightarrow$ power density radiated by the fields.

$\vec{P}$ describes the instantaneous power density carried by the wave

$\rightarrow$ Specialization to plane waves:

For a plane wave travelling along $\hat{n}$ $(\vec{k} = k\hat{n})$

$$\vec{H}_s = \frac{1}{\eta_c}\hat{n} \times \vec{E}_s \qquad \frac{\vec{E}_s}{\eta_c} = \vec{H}_s \times \hat{n}$$

$$\hat{n} = \frac{\vec{E}_s \times \vec{H}_s}{\eta_c} \rightarrow \vec{P} \text{ is along } \hat{n}.$$

Example:

Steady current I_c through a cylindrical resistor, radius b and length l. The conductivity of the resistor is σ.

$$\vec{J}_c = \hat{z} \cdot \frac{I_c}{\pi b^2}, \quad \vec{J}_c = \sigma \vec{E} \Rightarrow \vec{E} = \hat{z} \cdot \frac{I_c}{\pi b^2 \sigma}$$

$\sigma \neq \infty$ since conductor is not perfect.

$$\vec{H} = \frac{I_c}{2\pi b}\hat{\phi}$$

radial direction

$$\vec{P} = \vec{E} \times \vec{H} = (\hat{z} \times \hat{\phi}) \cdot \frac{I_c^2}{2\pi^2 b^3 \sigma} = -\hat{r} \cdot \frac{I_c^2}{2\pi^2 b^3 \sigma}$$

$-\int \vec{P} \cdot d\vec{s}$: total power flow.
$d\vec{s}$: surface element along $\hat{r}$

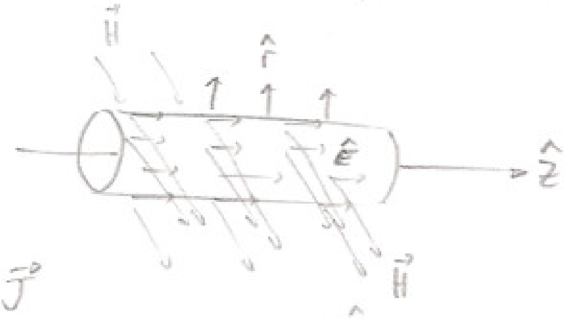
Total power flowing into resistor from the source of $\vec{E}, \vec{H}$ is

$$\left(\frac{I_c^2}{2\pi^2 b^3 \sigma}\right) \times (2\pi b l) = I_c^2 \frac{l}{\pi b^2 \sigma} = I_c^2 \cdot R.$$

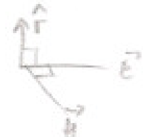
(regular multiplication)

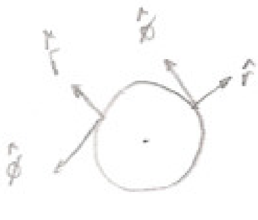
(R for cylinder length l radius b.)

power that heats resistor.



$\vec{E} = \sigma \vec{J}$





$\vec{P} = \vec{E} \times \vec{H}$ is the instantaneous power density

time-averaged $\vec{P}$ or $\vec{P}_{avg}$ or $\langle \vec{P} \rangle$

$$\vec{E}(z,t) = \mathrm{Re}\{ E_s e^{j\omega t} \}$$

$\vec{E}_s$ is complex but along $\hat{x}$.

$$\vec{E}_s = \hat{x}\, \mathrm{Re}\{ E_{re} + j E_{im} \}$$

$$\vec{H}_s = \hat{y} \left(\frac{E_{re} + j E_{im}}{\eta_c} \right)$$

$$\vec{E}(z,t) = \hat{x}\, \mathrm{Re}\{ (E_{re} + j E_{im})(\cos \omega t + j \sin \omega t) \}$$

$$= \hat{x}\, \mathrm{Re}\{ E_{re} \cos \omega t - E_{im} \sin \omega t + j(\cdots\cdots) \}$$

$$\vec{E}(z,t) = \hat{x} \left(E_{re} \cos \omega t - E_{im} \sin \omega t \right)$$

$$\vec{H}(z,t) = \hat{y}\, \mathrm{Re}\left\{ \left(\frac{E_{re} + j E_{im}}{\eta_c} \right) \cdot e^{j\omega t} \right\}$$

$$\frac{E_r + jE_i}{\eta_c} = H_r + jH_i \qquad \left(H_r \neq \frac{E_r}{\eta_c} \right)$$

$$\vec{H}(z,t) = \hat{y} \left(H_{re} \cos \omega t - H_{im} \sin \omega t \right)$$

$$\vec{P} = (\vec{E} \times \vec{H}) = (\hat{x} \times \hat{y}) \cdot \left[E_{re} H_{re} \cos^2 \omega t + E_{im} H_{im} \sin^2 \omega t - (E_{re} H_{im} + E_{im} H_{re})(\cos \omega t \sin \omega t) \right]$$

Taking time average:

$$\langle \vec{P} \rangle = \frac{1}{T} \int_0^T \vec{P}\, dt$$

$$\Rightarrow \cos^2 \omega t \rightarrow \frac{1}{2}, \quad \sin^2 \omega t \rightarrow \frac{1}{2}, \quad \sin \omega t \cos \omega t = 0$$

$$\langle \vec{P} \rangle = \hat{z} \frac{1}{2} \left[E_{re} H_{re} + E_{im} H_{im} \right] = \frac{1}{2} \mathrm{Re}\{ \vec{E}_s \times \vec{H}_s^* \}$$ (can be verified by direct substitution)

$$\rightarrow \boxed{\langle P \rangle = \frac{1}{2} \mathrm{Re}\{ \vec{E}_s \times \vec{H}_s^* \}}$$

NOTE: $\vec{E}_s = \hat{x}\, e^{-\alpha z} e^{-j\beta z} E_{0x}$, E_{0x} is real

$$\vec{H}_s = \hat{y}\, e^{-\alpha z} e^{-j\beta z} \frac{E_{0x}}{\eta_c}$$

Write $\eta_c = |\eta_c| e^{j\theta_\eta}$

$$\vec{H}_s^{\,*} = \hat{y}\, e^{-\alpha z} e^{j\beta z} \frac{E_{0x}}{|\eta_c|} e^{j\theta_\eta}$$

$$\vec{E}_s \times \vec{H}_s^{\,*} = (\hat{x}\times\hat{y})\, e^{-2\alpha} \frac{E_{0x}^2}{|\eta_c|} e^{j\theta_\eta}$$

$$\langle\vec{P}\rangle = \frac{1}{2} \cdot \mathrm{Re}\left\{ \vec{E}_s \times \vec{H}_s^{\,*} \right\} = \hat{z}\,\frac{1}{2} e^{-2\alpha} \frac{E_{0x}^2}{|\eta_c|} \cos\theta_\eta .$$